ON THE UNSTEADY MOTIONS OF A HEAVY FLUID AT A SLOPING BEACH

(O NEUSTANOVIVSHIKHSIA DVIZHENIIAKH TIAZHELOI ZHIDKOSTI U NAKLONNOGO BEREGA)

PMM Vol. 24, No. 3, 1960, pp. 554-557

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(Received 26 January 1960)

Waves on the surface of an infinitely deep fluid, and also on the surface of a fluid of finite constant depth, have been investigated in many papers. In this connection Fourier transformations were used for the most part for the solution to the problem.

We note that in [1] and [2] a system of standing waves at beaches of one surface slope were investigated. In the work of Keldysh [3] a solution of the non-stationary problem for a beach with a slope angle of 45° was obtained with the help of an integration with respect to a parameter of the solution of the standing wave problem and the construction of an inversion formula.

In an analogous manner the linear problem of the unsteady motions of an incompressible fluid at a sloping beach is considered below. A simpler inversion formula is obtained with the help of the generalized Fourier transformations. As an example the problem of Cauchy-Poisson waves at a beach with a slope angle of 45° and also with a small slope in the shallow water approximation is examined.

1. We will consider the planar case. Let the fluid be bounded by the free surface which coincides with the x-axis and by a rigid wall which makes an angle β with it. We will consider that the motion begins from a state of rest. Therefore, the velocity potnetial $\phi(x, y, t)$, which satisfies the equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{1.1}$$

in the region of flow, exists.

On the free surface ϕ satisfies the equation

$$\frac{\partial\varphi}{\partial y} = -\frac{1}{g}\frac{\partial^2\varphi}{\partial t^2} + \frac{1}{\rho g}\frac{\partial f_0(x, 0; t)}{\partial t} \quad \textbf{for } y = 0 \tag{1.2}$$

where f_0 is a known function which describes the pressure distribution on the surface as a function of time. On the rigid wall the condition of no through flow

$$\partial\varphi/\partial n = 0 \tag{1.3}$$

must be satisfied, where n is the direction normal to the rigid wall. The function ϕ can satisfy initial conditions

$$\varphi(x, 0; 0) = \frac{1}{\rho} F(x), \qquad \frac{\partial\varphi(x, 0; 0)}{\partial t} = gf(x) \tag{1.4}$$

2. Henceforth we will consider only the problem of Cauchy-Poisson waves with the initial conditions (1.4) and we will assume that for $t > 0$ no external pressure forces act on the fluid ($f_0 \equiv 0$ in Expression (1.2)). As is known from general theory, it is sufficient to solve the problem for the case $f(x) \equiv 0$. We will seek a solution of Equation (1.1) in the following form:

$$\varphi(x, y; t) = \int_0^\infty F_1(m) \cos\sqrt{mg}\, t\, \Phi(x, y; m)\, dm \tag{2.1}$$

where $F_1(m)$ is an unknown function and $\Phi(x, y; m)$ is a function, harmonic within the angle β, which satisfies the conditions

$$\Phi_y - m\Phi = 0 \text{ при } \quad y = 0,\ x > 0, \qquad \frac{\partial\Phi}{\partial n} = 0 \textbf{ on the solid boundary} \tag{2.2}$$

The problem of finding the function $\Phi(x, y; m)$ for the Conditions (2.2) has been solved [2] for the angles $\beta = \pi/2n$, where n is an integer. For $n = 2$ this solution has the form

$$\Phi(x, y; m) = \frac{\pi}{\sqrt{2}} \operatorname{Re}\left[e^{\frac{1}{4}i\pi} e^{-m(x+iy)} + e^{-\frac{1}{4}i\pi} e^{-im(x+iy)}\right] \tag{2.3}$$

We will choose the function $F_1(m)$ so that $\phi(x, y; t)$, defined by Formula (2.1), satisfies the initial Condition (1.4). Substituting (2.1) into (1.4), we obtain

$$\frac{1}{\rho} F(x) = \int_0^\infty F_1(m)\, \Phi(x, 0; m)\, dm \tag{2.4}$$

Thus, for determining the function $F_1(m)$, a Fredholm integral equation of the first kind with the kernel $\Phi(x, 0; m)$ is obtained. Its solution for arbitrary n has a complicated form [4]. Therefore, we will confine our attention to the case $n = 2$, i.e. to a slope angle of the beach equal to 45° (for $n = 1$ (2.4) becomes the usual Fourier transformation and the problem reduces to the reflection of waves from a vertical beach or to the problem of Cauchy-Poisson waves on the surface of an infinitely deep fluid). In this case (2.4) has the form

$$\frac{1}{\rho} F(x) = \frac{\pi}{2} A \int_0^\infty F_1(m)(e^{-mx} + \cos mx - \sin mx)\, dm \tag{2.5}$$

where A is for the present an undetermined constant.

We will show that the kernel of Equation (2.5) is a Fourier kernel. For this it is sufficient [4] to establish that it satisfies the functional equation

$$K(s) K(1-s) = 0 \qquad \left(K(s) = \int_0^\infty \varphi(x) x^{s-1} dx\right) \tag{2.6}$$

Here K is the Mellin transformation of the function $\phi(x)$. Substituting the expression $\phi(x) = 1/2\,\pi\, A(c^{-x} + \cos x - \sin x)$ into (2.6), we prove to ourselves that this equation is satisfied for the value of the constant $A = 2/\pi\sqrt{\pi}$. Consequently, in this case the Expression (2.5) represents the generalized Fourier transformation and its inversion has the symmetrical form

$$F_1(m) = \frac{1}{\rho\sqrt{\pi}} \int_0^\infty F(x)(e^{-mx} + \cos mx - \sin mx)\, dx \tag{2.7}$$

Substituting (2.7) into (2.1), we obtain the solution of the problem of Cauchy-Poisson waves (for the angle $\beta = 45°$)

$$\varphi(x, y; t) = \frac{1}{\pi\rho} \int_0^\infty \cos \sqrt{mg}t\, [e^{-mx}(\cos my + \sin my) + e^{my}(\cos mx - \sin mx)] \times$$

$$\times \int_0^\infty (e^{-mx} + \cos mx - \sin mx) F(x)\, dx dm \tag{2.8}$$

3. We shall pass on to the consideration of special cases. Let $F(x) = J\delta(x)$, where $\delta(x)$ is the Dirac delta-function. This corresponds to an instantaneous impulse in the quantity J which is applied in the neighborhood of the origin of the coordinate system. Formula (2.8) in this case acquires the form

$$\varphi(x, y; t) = \frac{2J}{\pi\rho} \int_0^\infty \cos \sqrt{mg}\, t\, [e^{-mx}(\cos my + \sin my) + e^{my}(\cos mx - \sin mx)]\, dm \tag{3.1}$$

Multiplying (3.1) by $1/g$, differentiating with respect to t and assuming $y = 0$, we obtain the elevation of the free surface

$$\eta(x, t) = -\frac{2J}{\pi\rho g} \int_0^\infty \sqrt{mg} \sin \sqrt{mg}\, t (e^{-mx} + \cos mx - \sin mx) dm \tag{3.2}$$

By the same arguments it is easy to obtain the solution which corresponds to an initial elevation of the free surface concentrated near the origin of the coordinate system. Designating by Q the volume of fluid contained between the profile of the initial elevation and the x-axis, we obtain expressions for and

$$\varphi(x, y; t) = \frac{2Q\sqrt{g}}{\pi}\int_0^\infty \sin\sqrt{mg}\,t\,[e^{-mx}(\cos my + \sin my) + e^{my}(\cos mx - \sin mx)]\frac{dm}{\sqrt{m}} \quad (3.3)$$

$$\eta(x, t) = \frac{2Q}{\pi}\int_0^\infty \cos\sqrt{mg}\,t\,(e^{-mx} + \cos mx - \sin mx)dm \quad (3.4)$$

The integrals obtained can not be evaluated in elementary functions. We will calculate the value of (3.4) at large distances from the origin of the coordinate system. Expanding the integrand into a series and carrying out the integration, we obtain

$$\eta(x, t) = \frac{4Q}{\pi x}\left(\frac{\omega}{1!!} - \frac{\omega^2}{3!} + \frac{\omega^5}{9!!} - \frac{\omega^6}{11!!} + \frac{\omega^9}{17!!} - \frac{\omega^{10}}{19!!} + \ldots\right) \quad (3.5)$$

where $\omega = gt^2/2x$. To obtain a formula which describes the form of the surface for the case of an initial impulse, it is necessary to calculate the integral (3.1) or (3.2). However, it is easy to see that this can also be obtained directly by differentiating (3.5) with respect to t and multiplying the expression obtained by $J/Q\rho g$. The power series (3.5) gives good agreement only for small values of ω. An asymptotic expression for η can be obtained for large ω. Evaluating the integral by the method of stationary phase, we find

$$\eta = \frac{2Q}{x\sqrt{\pi}}\sqrt{\frac{\omega}{2}}\left[\cos\left(\frac{\omega}{2} - \frac{\pi}{4}\right) + \sin\left(\frac{\omega}{2} - \frac{\pi}{4}\right)\right]$$

4. We will consider the problem of Cauchy-Poisson waves arising from an initial elevation of the surface at a distance x_1 from the origin of the coordinate system. In this case

$$F(x) = Q\delta(x - x_1)$$

Using formula (2.8), we obtain an expression for $\eta(x, t)$

$$\eta(x, t) = \frac{2Q}{\pi}\int_0^\infty \cos\sqrt{mg}\,t\,(e^{-mx} + \cos mx - \sin mx)(e^{-mx_1} + \cos mx_1 - \sin mx_1)dm \quad (4.1)$$

An analogous formula is obtained for the case of an intial impulse in the quantity J applied at the point $(x_1, 0)$. The integral in (4.1) converges very slowly. Therefore, for making calculations with this formula it is necessary to use computing machines.

To get an idea of the phenomenon of the origin of the waves and their reflection from the sloping beach, a calculation was carried out on the computing machine "Arrow" with the aid of Formula (4.1) for values of $g = 9.81$ m/sec^2 and for $t = 0.5$, 4 and 8 sec. The initial elevation was given in the form of a step function equal to zero for $x < 1$ m and for $x > 1.1$ m and equal to unity for 1 m $< x <$ 1.1 m.

The results of the caluclation are represented in the form of graphs in Fig. 1, where $\eta_1 = \pi\eta/2Q$. An analogous calculation was performed for the case of an initial impulse of the same form for the same values of the parameters. The corresponding graphs have been constructed in Fig. 2,

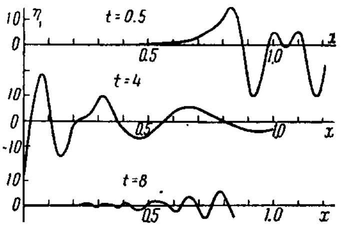

Fig. 1.

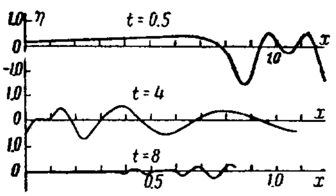

Fig. 2.

where $\eta_1 = \pi \rho g\eta/2J$.

We will obtain the asymptotic expression for (4.1) for large values of the quantity gt^2 and for $x < x_1$. Applying the method of stationary phase, we find

$$\eta(x, t) = Q\sqrt{\frac{g}{\pi}}\, t\left\{\frac{1}{\sqrt{(x_1-x)^3}}\cos\frac{1}{4}\left(\frac{gt^2}{x_1-x}-\pi\right)+\frac{1}{\sqrt{(x_1+x)^3}}\sin\frac{1}{4}\left(\frac{gt^2}{x_1+x}-\pi\right)+\right.$$
$$+\frac{1}{x^{3/2}}\exp\left(-\frac{gt^2x_1}{4x^2}\right)\left[\cos\frac{1}{4}\left(\frac{gt^2}{x}-\pi\right)-\sin\frac{1}{4}\left(\frac{gt^2}{x}-\pi\right)\right]+$$
$$\left.+\frac{1}{x_1^{3/2}}\exp\left(-\frac{gt^2x}{4x_1^2}\right)\left[\cos\frac{1}{4}\left(\frac{gt^2}{x_1}-\pi\right)-\sin\frac{1}{4}\left(\frac{gt^2}{x_1}-\pi\right)\right]\right\} \quad (4.2)$$

5. If the slope angle of the beach is small ($< 6°$ according to [2]), then shallow wave theory can be applied at not very large distances from the origin of the coordinate system. In the work of [2] there is obtained the corresponding system of standing waves

$$\varphi(x, 0; t) = A\cos\sqrt{mg}\, tJ_0(2\sqrt{mx/q}) \quad (5.1)$$

where q is the slope angle of the beach and J_0 is the Bessel function of zero order. We will construct a more general solution, multiplying (5.1) by the arbitrary function $F_1(\sqrt{m})$ and integrating with respect to m from 0 to ∞

$$\varphi(x, 0; t) = \int_0^\infty \cos \sqrt{mg}\, t\, J_0(2\sqrt{mx/q})\, F_1(\sqrt{m})\, dm \tag{5.2}$$

Setting $t = 0$ in (5.2), making the substitution of variables $\sqrt{m} = \sigma$ and using the initial condition (1.4), we obtain an integral equation for determining $F_1(\sigma)$

$$\frac{1}{\rho} F(x) = 2\int_0^\infty F_1(\sigma) J_0\left(2\sigma\sqrt{\frac{x}{q}}\right) \sigma d\sigma$$

As we see, $F(x)$ is the Hankel transformation of zero order of the function $F_1(\sigma)$. Using the inversion formula and substituting in (5.2), we obtain

$$\varphi(x, 0; t) = \frac{1}{\rho q}\int_0^\infty \cos \sigma \sqrt{g}\, t J_0\left(2\sigma\sqrt{\frac{x}{q}}\right) \sigma d\sigma \int_0^\infty F(x) J_0\left(2\sigma\sqrt{\frac{x}{q}}\right) \sqrt{x}\, d\sqrt{x} \tag{5.3}$$

We shall calculate the elevation of the free surface which arises under the action of the initial impulse of the quantity J, concentrated in the neighborhood of the origin of the coordinate system. From (5.3) with $F(x) = J\delta(x)$ we find

$$\eta(x, t) = -\frac{4\Gamma(^3/_2) J}{\sqrt{\pi g}\, \rho q\, x^{3/2}} \left(\frac{gt^2}{x} + \frac{2}{q}\right)\left(\frac{gt^2}{x} - \frac{4}{q}\right)^{-5/2} \tag{5.4}$$

The corresponding formula for the case in which the impulse is applied at a distance x_1 from the origin of the coordinate system has the form

$$\varphi(x, 0; t) = \frac{J}{\rho q}\int_0^\infty \cos \sigma \sqrt{g}\, t J_0\left(2\sigma\sqrt{\frac{x}{q}}\right) J_0\left(2\sigma\sqrt{\frac{x_1}{q}}\right) \sigma d\sigma \tag{5.5}$$

It is easy to calculate the displacement of the surface at the origin of the coordinate system as a function of time. Assuming $x = 0$ in (5.5), we satisfy ourselves that the formula coincides in form with (5.3) for the case considered above. Consequently, the desired dependence is given by Formula (5.4), where it is necessary to replace x by x_1.

In conclusion I want to express deep gratitude to N.N. Moiseef for valuable advice and also to take this opportunity to thank E.P. Borisov for assistance in carrying out the computer calculations.

BIBLIOGRAPHY

1. Lewy, H., Water waves on sloping beaches. *Bulletin of the American Mathematical Society* Vol. 52, 1946.

2. Stoker, J.J., Surface waves in water of variable depth. *Quarterly of Applied Mathematics* Vol. 5, 1-54, 1947.

3. Keldysh, M.V., K zadache ob otrazhenii voln na poverkhnosti tiazheloi zhidkosti (On the problem of the reflection of waves on the surface of a heavy fluid). *Tekhn. zametki TsAGI* No. 52, Issue 2, 3-4, 1935.

4. Titchmarsh, E., Vvedenie v teoriiu integralov Fur'e (Introduction to the theory of Fourier integrals). OGIZ, Moscow-Leningrad, 1948.

Translated by R.D.C.